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# Three-point functions in sine-Liouville theory

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Abstract: We calculate the three-point functions in the sine-Liouville theory explicitly. The same calculation was done in the (unpublished) work of Fateev, Zamolodchikov and Zamolodchikov to check the conjectured duality between the sine-Liouville and the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ coset CFTs. The evaluation of correlators boils down to that of a free-field theory with a certain number of insertion of screening operators. We prove that the winding number conservation is violated up to $\pm 1$ in three-point functions, which is in agreement with the result of FZZ that in generic $N$-point correlators the winding number conservation is violated up to $N-2$ units. A new integral formula of Dotsenko-Fateev type is derived, using which we write down the generic three-point functions of tachyons explicitly. When the winding number is conserved, the resultant expression is shown to reproduce the correlators in the coset model correctly, including the group-theoretical factor. As an application, we also study the superstring theory on linear dilaton background which is described by super-Liouville theory. We obtain the three-point amplitude of tachyons in which the winding number conservation is violated.

Keywords: Field Theories in Lower Dimensions, Conformal Field Models in String Theory, Black Holes in String Theory.

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## Contents





## 1. Introduction

The $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ coset model has a number of applications in recent topics in string theory. As was first studied in [1, 2], the model is known to describe a certain two-dimensional black hole, and serve as an example of toy models in which one can analyze the quantum nature of black holes using conventional conformal field theory. A matrix model for this black hole has also been proposed recently [3]. It has also been used to analyze the theory of NS5-branes [4], or strings propagating in the vicinity of singularities of Calabi-Yau manifolds [5]–[9] in the spirit of holographic duality. The string theory on these backgrounds are supposed to be described by linear dilaton CFTs, but typically the dilaton diverges at the points where the NS5-branes or the singularity of Calabi-Yau manifolds are located. The use of the coset CFT therefore serves as a regularization. The coset model also has an application to the analysis of the $\mathrm{SL}(2,\mathbb{R})$ WZW model, where the processes involving “long strings” have been studied in recent works [10]–[13].

Fateev, Zamolodchikov and Zamolodchikov(FZZ) have made a conjecture that the above coset model has a dual description in terms of the sine-Liouville theory [14]. They checked the conjecture from various approaches, including the explicit comparison of two- and three-point functions in both theories. Based on this duality, there have been a lot of analysis of string theory on singular Calabi-Yau manifolds, using the supersymmetric version of sine-Liouville theory to describe the deformed singularities. It should also be noted that the matrix model [3] has also been proposed based upon this conjecture. So it is of great importance to know this duality in detail. Recently, [15] has made an analysis using the linear sigma model techniques and concluded that the supersymmetric version of these two theories are dual to each other. To analyze the coset model from the dual sine-Liouville theory, it would also be very useful if we have explicit expressions for correlators in the latter theory.

In this paper we study the three-point functions in sine-Liouville theory on a sphere. In a perturbative approach they can be calculated as free-field correlators with a number of insertions of "screening operators". So we can express them as certain multiple integrals over the complex plane. However, as we will see, to evaluate the integral expression we will need to develop some techniques and have a new formula like the one in [16].

This paper is organized as follows. Section 2 is a brief introduction to sine-Liouville theory, in which we shall give an integral expression for generic three-point functions of tachyons. In section 3 we undertake a detailed analysis of the resultant integral expression. We utilize the techniques of [16, 17] for translating $d^2z$ integrals into products of contour integrals, and evaluating the resultant contour integrals. Along the way we prove that the three-point correlator vanishes unless the total winding number is either 0 or $\pm 1$. This is in consistency with the result of FZZ that in generic $N$-point functions the winding number conservation is violated up to $N-2$ units. We finally end up with some new formulae and closed analytic expressions for generic three-point functions of tachyons. Section 4 compares our expression for the correlators in sine-Liouville theory with the known formula in the coset model. In the case where the total winding number is zero, we see that the two expressions are in good agreement, and the $\mathrm{SL}(2,\mathbb{R})$ group-theoretical factor is correctly reproduced in sine-Liouville theory. Using our new formulae, in section 5 we analyze the three-point amplitudes in superstring theory on a linear-dilaton background which is described by the $N=2$ super-Liouville theory. The last section concludes with a brief discussion. Our new formulae are summarized in the appendix for the readers' use.

## 2. Sine-Liouville theory

Consider the two-dimensional free CFT with the action

$$S = \frac{1}{2\pi}\int d^2z\left(\partial x\bar{\partial}x + \partial\rho\bar{\partial}\rho - \frac{\rho}{2\sqrt{2k'}}\sqrt{g}R\right), \tag{2.1}$$

with $k' = k - 2$. If we introduce the interaction of the form

$$S_{\rm int} = \frac{\lambda}{2\pi} \int d^2 z e^{-\sqrt{k'/2}\,\rho} \cos\sqrt{\frac{k}{2}}(x_L - x_R) \tag{2.2}$$

which is made of the screening currents

$$\eta^{\pm} = \exp\left[\pm i\sqrt{\frac{k}{2}}\,x - \sqrt{\frac{k'}{2}}\,\rho\right], \tag{2.3}$$

the resultant theory is conjectured to give a dual description of the $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ coset model. In this conjecture, the primary fields are described by tachyonic vertex operators

$$T_{jm\bar{m}}(z,\bar{z}) = T_{jm}(z)T_{j\bar{m}}(\bar{z}) = \exp\left[j\sqrt{\frac{2}{k'}}\rho + i\sqrt{\frac{2}{k}}\,(m x_L + \bar{m} x_R)\right], \tag{2.4}$$

satisfying the quantization condition

$$m = \frac{1}{2}(p + kw)\,, \qquad \bar{m} = \frac{1}{2}(p - kw)\,. \tag{2.5}$$

The integers $p$ and $w$ are the momentum and the winding number along the periodic direction of the euclidean two-dimensional black hole (cigar). It is expected that in generic correlation functions in the coset model the winding number do not conserve, while the momentum does conserve. It is therefore interesting to study how this property is realized in the dual sine-Liouville theory. It involves the calculation of correlation functions in free CFT of the form:

$$\frac{1}{m!n!}\left\langle \prod_a T_{j_a m_a \bar{m}_a}(w_a)\left(\int d^2 z S^{+}_{\rm int}(z)\right)^m \left(\int d^2 z S^{-}_{\rm int}(z)\right)^n\right\rangle \tag{2.6}$$

where the two screening operators $S_{\mathrm{int}^\pm}$ are defined by

$$S^{+}_{\rm int}(z,\bar{z}) = \eta^{+}_{L}(z)\eta^{-}_{R}(\bar{z})\,, \qquad S^{-}_{\rm int}(z,\bar{z}) = \eta^{-}_{L}(z)\eta^{+}_{R}(\bar{z})\,. \tag{2.7}$$

We are particularly interested in three-point functions on the sphere. From the momentum conservation law of free CFT with background charge we have

$$\sum_{a=1}^{3} j_a + 1 = \frac{k'}{2}(n+m)\,, \qquad \sum_{a=1}^{3} m_a = \frac{k}{2}(n-m)\,, \qquad \sum_{a=1}^{3} \bar{m}_a = \frac{k}{2}(m-n)\,, \tag{2.8}$$

in accord with the momentum conservation and the winding number non-conservation,

$$\sum p_a = 0\,, \qquad \sum w_a = n - m\,. \tag{2.9}$$

Using the free field OPEs we evaluate (2.6) and obtain

$$\begin{aligned} I = & \frac{1}{m!n!}\prod_{a<b} w_{ab}^{\frac{2m_a m_b}{k}-\frac{2j_a j_b}{k'}}\bar{w}_{ab}^{\frac{2\bar{m}_a\bar{m}_b}{k}-\frac{2j_a j_b}{k'}}\times \\ & \times \prod_a \int d^{2m}z_i d^{2n}z_{\hat{i}} \prod_{i<j}^m |z_{ij}|^2 \prod_{\hat{i}<\hat{j}}^n |z_{\hat{i}\hat{j}}|^2 \prod_{i,\hat{j}}^{m,n} |z_{i\hat{j}}|^{2-2k} \times \\ & \times \prod_a \left[ \prod_{i=1}^m (z_i - w_a)^{j_a+m_a}(\bar{z}_i - \bar{w}_a)^{j_a-\bar{m}_a} \prod_{\hat{i}=\hat{1}}^{\hat{n}} (z_{\hat{i}} - w_a)^{j_a-m_a}(\bar{z}_{\hat{i}} - \bar{w}_a)^{j_a+\bar{m}_a} \right]. \end{aligned} \tag{2.10}$$

For three-point functions, the $w_a$-dependence is determined from the conformal invariance as

$$I = \prod_{a<b}^3 w_{ab}^{-h_{ab}}\bar{w}_{ab}^{-\bar{h}_{ab}} D(j_a, m_a, \bar{m}_a)\,, \tag{2.11}$$

where we introduced the notation such as $h_{12} = h_1+h_2-h_3$. $h_a$ denotes the conformal weight of $T_{j_a m_a \bar{m}_a}$ which is given by

$$h = -\frac{j(j+1)}{k'} + \frac{m^2}{k}\,, \qquad \bar{h} = -\frac{j(j+1)}{k'} + \frac{\bar{m}^2}{k}\,. \tag{2.12}$$

Then all we need is the integral formula of the form:

$$\begin{aligned} I(\alpha,\beta,\gamma) = & \frac{1}{m!n!}\int \prod_{i=1}^m \left(d^2 z_i z_i^{\alpha}\bar{z}_i^{\bar{\alpha}}(z_i-1)^{\beta}(\bar{z}_i-1)^{\bar{\beta}}\right) \prod_{i<j}^m |z_{ij}|^2 \times \\ & \times \prod_{\hat{i}=\hat{1}}^n \left(d^2 z_{\hat{i}} z_{\hat{i}}^{\alpha'}\bar{z}_{\hat{i}}^{\bar{\alpha}'}(z_{\hat{i}}-1)^{\beta'}(\bar{z}_{\hat{i}}-1)^{\bar{\beta}'}\right) \prod_{\hat{i}<\hat{j}}^n |z_{\hat{i}\hat{j}}|^2 \prod_{i,\hat{j}}^{m,n} |z_{i\hat{j}}|^{2-2k}\,. \end{aligned} \tag{2.13}$$

This must be symmetric in $\alpha, \beta$ and $\gamma$ obeying the relation

$$\begin{aligned} \alpha+\beta+\gamma &= \bar{\alpha}+\bar{\beta}+\bar{\gamma} = kn-m-n-1\,, \\ \alpha'+\beta'+\gamma' &= \bar{\alpha}'+\bar{\beta}'+\bar{\gamma}' = km-m-n-1\,, \end{aligned} \tag{2.14}$$

reflecting the symmetry under the exchange of three vertices. Once the above integral is calculated, the three-point functions can be obtained by the replacement

$$\begin{aligned} &\alpha = j_1+m_1\,, && \alpha' = j_1-m_1\,, && \bar{\alpha} = j_1-\bar{m}_1\,, && \bar{\alpha}' = j_1+\bar{m}_1\,, \\ &\beta = j_2+m_2\,, && \beta' = j_2-m_2\,, && \bar{\beta} = j_2-\bar{m}_2\,, && \bar{\beta}' = j_2+\bar{m}_2\,, \\ &\gamma = j_3+m_3\,, && \gamma' = j_3-m_3\,, && \bar{\gamma} = j_3-\bar{m}_3\,, && \bar{\gamma}' = j_3+\bar{m}_3. \end{aligned} \tag{2.15}$$



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# 3. Integral formula

Here we shall give the formula for the integral

$$I = \frac{1}{m!n!} \int \prod_{i=1}^{m} \left( d^2 z_i z_i^{\alpha} \bar{z}_i^{\bar{\alpha}} (1 - z_i)^{\beta} (1 - \bar{z}_i)^{\bar{\beta}} \right) \prod_{i<j}^{m} |z_{ij}|^{4\rho} \times$$

$$\times \prod_{\hat{i}=\hat{1}}^{n} \left( d^2 z_{\hat{i}} z_{\hat{i}}^{\alpha'} \bar{z}_{\hat{i}}^{\bar{\alpha}'} (1 - z_{\hat{i}})^{\beta'} (1 - \bar{z}_{\hat{i}})^{\bar{\beta}'} \right) \prod_{\hat{i}<\hat{j}}^{n} |z_{\hat{i}\hat{j}}|^{4\rho'} \prod_{i,\hat{j}}^{m,n} |z_{i\hat{j}}|^{4\sigma} , \tag{3.1}$$

with $\rho = \rho' = 1/2$.

## 3.1 Decomposing into two contour integrals

The first thing to do is to transform it into a multiple contour integral. To do this, we rewrite the above expression in terms of real variables $(x_i, y_i, x_{\hat{i}}, y_{\hat{i}})$

$$I = \frac{1}{m!n!} \int \prod_{i=1}^{m} \left[ dx_i dy_i (x_i + iy_i)^{\alpha} (x_i - iy_i)^{\bar{\alpha}} (1 - x_i - iy_i)^{\beta} (1 - x_i + iy_i)^{\bar{\beta}} \right] \times$$

$$\times \prod_{\hat{i}=\hat{1}}^{n} \left[ dx_{\hat{i}} dy_{\hat{i}} (x_{\hat{i}} + iy_{\hat{i}})^{\alpha'} (x_{\hat{i}} - iy_{\hat{i}})^{\bar{\alpha}'} (1 - x_{\hat{i}} - iy_{\hat{i}})^{\beta'} (1 - x_{\hat{i}} + iy_{\hat{i}})^{\bar{\beta}'} \right] \times$$

$$\times \prod_{i<j}^{m} (x_{ij}^2 + y_{ij}^2)^{2\rho} \prod_{\hat{i}<\hat{j}}^{n} \left( x_{\hat{i}\hat{j}}^2 + y_{\hat{i}\hat{j}}^2 \right)^{2\rho'} \prod_{i,\hat{j}}^{m,n} \left( x_{i\hat{j}}^2 + y_{i\hat{j}}^2 \right)^{2\sigma} , \tag{3.2}$$

Next we rotate the integration contours of $y_i$ and $y_{\hat{i}}$ by $-(90 - \epsilon)$ degrees. This can be done by introducing new variables

$$y_i = -i(1 + i\epsilon)\tilde{y}_i , \qquad y_{\hat{i}} = -i(1 + i\epsilon)\tilde{y}_{\hat{i}}$$

and make the change of the integration variables. Then we get (omitting tildes)

$$I = \frac{1}{m!n!} \int \prod_{i=1}^{m} \Big[ -idx_i dy_i (x_i + y_i + i\epsilon y_i)^{\alpha} (x_i - y_i - i\epsilon y_i)^{\bar{\alpha}} \times$$

$$\times (1 - x_i - y_i - i\epsilon y_i)^{\beta} (1 - x_i + y_i + i\epsilon y_i)^{\bar{\beta}} \Big] \times$$

$$\times \prod_{\hat{i}=1}^{\hat{n}} \Big[ -idx_{\hat{i}} dy_{\hat{i}} (x_{\hat{i}} + y_{\hat{i}} + i\epsilon y_{\hat{i}})^{\alpha} (x_{\hat{i}} - y_{\hat{i}} - i\epsilon y_{\hat{i}})^{\bar{\alpha}} \times$$

$$\times (1 - x_{\hat{i}} - y_{\hat{i}} - i\epsilon y_{\hat{i}})^{\beta} (1 - x_{\hat{i}} + y_{\hat{i}} + i\epsilon y_{\hat{i}})^{\bar{\beta}} \Big] \times \tag{3.3}$$

$$\times \prod_{i<j}^{m} \left[ x_{ij}^2 - (1+i\epsilon) y_{ij}^2 \right]^{2\rho} \prod_{\hat{i}<\hat{j}}^{\hat{n}} \left[ x_{\hat{i}\hat{j}}^2 - (1+i\epsilon) y_{\hat{i}\hat{j}}^2 \right]^{2\rho'} \prod_{i,\hat{j}}^{m,\hat{n}} \left[ x_{i\hat{j}}^2 - (1+i\epsilon) y_{i\hat{j}}^2 \right]^{2\sigma} ,$$

where all the variables take values on the real line. We further introduce new integration variables

$$z_i = x_i + y_i\,, \qquad w_i = x_i - y_i\,, \qquad z_{\hat{i}} = x_{\hat{i}} + y_{\hat{i}}\,, \qquad w_{\hat{i}} = x_{\hat{i}} - y_{\hat{i}}\,.$$

to rewrite the integral as follows:

$$\begin{aligned} I = \frac{1}{m!n!} \int \prod_{i=1}^{m} \Bigg[ \frac{idz_i dw_i}{2} \left(z_i - i\epsilon w_i\right)^{\alpha} \left(w_i - i\epsilon z_i\right)^{\bar{\alpha}} \times \\ \times \left(1 - z_i - i\epsilon(1-w_i)\right)^{\beta} \left(1 - w_i - i\epsilon(1-z_i)\right)^{\bar{\beta}} \Bigg] \times \\ \times \prod_{\hat{i}=\hat{1}}^{n} \Bigg[ \frac{idz_{\hat{i}} dw_{\hat{i}}}{2} \left(z_{\hat{i}} - i\epsilon w_{\hat{i}}\right)^{\alpha'} \left(w_{\hat{i}} - i\epsilon z_{\hat{i}}\right)^{\bar{\alpha'}} \times \\ \times \left(1 - z_{\hat{i}} - i\epsilon(1-w_{\hat{i}})\right)^{\beta'} \left(1 - w_{\hat{i}} - i\epsilon(1-z_{\hat{i}})\right)^{\bar{\beta'}} \Bigg] \times \\ \times \prod_{i<j}^{m} \left[ \left(z_{ij} - i\epsilon w_{ij}\right)^{2\rho} \left(w_{ij} - i\epsilon z_{ij}\right)^{2\rho} \right] \prod_{\hat{i}<\hat{j}}^{n} \left[ \left(z_{\hat{i}\hat{j}} - i\epsilon w_{\hat{i}\hat{j}}\right)^{2\rho'} \left(w_{\hat{i}\hat{j}} - i\epsilon z_{\hat{i}\hat{j}}\right)^{2\rho'} \right] \times \\ \times \prod_{i,\hat{j}}^{m,n} \left[ \left(z_{i\hat{j}} - i\epsilon w_{i\hat{j}}\right)^{2\sigma} \left(w_{i\hat{j}} - i\epsilon z_{i\hat{j}}\right)^{2\sigma} \right], \end{aligned} \tag{3.4}$$

where all the variables takes values on the real line. The integral is "almost factorized" into two multiple contour integrals. The appearance of $\epsilon$ serves as the specification of how to arrange the contours. To see how the contours should be arranged, suppose that we are to calculate the above integral by first performing the integration over $w_i$ and $w_{\hat{i}}$ with $(z_i, z_{\hat{i}})$ held fixed. So we first focus on the following part

$$\begin{aligned} \int d^{m+n}w \prod_{i=1}^{m} \left[ \left(w_i - i\epsilon z_i\right)^{\bar{\alpha}} \left(1 - w_i - i\epsilon(1-z_i)\right)^{\bar{\beta}} \right] \times \\ \times \prod_{\hat{i}=\hat{1}}^{n} \left[ \left(w_{\hat{i}} - i\epsilon z_{\hat{i}}\right)^{\bar{\alpha'}} \left(1 - w_{\hat{i}} - i\epsilon(1-z_{\hat{i}})\right)^{\bar{\beta'}} \right] \times \\ \times \prod_{i<j}^{m} \left[ \left(w_{ij} - i\epsilon z_{ij}\right)^{2\rho} \right] \prod_{\hat{i}<\hat{j}}^{n} \left[ \left(w_{\hat{i}\hat{j}} - i\epsilon z_{\hat{i}\hat{j}}\right)^{2\rho'} \right] \prod_{i,\hat{j}}^{m,n} \left[ \left(w_{i\hat{j}} - i\epsilon z_{i\hat{j}}\right)^{2\sigma} \right]. \end{aligned} \tag{3.5}$$

The factors in the first two lines determine how to deform the contours of $w_i$ and $w_{\hat{i}}$ to keep away from the singularities 0 and 1. For example, if $0 < z_1 < 1$, then the contour of $w_1$ should pass below 0 and above 1. The rule is summarized in the figure 1. For more explanation of this manipulation of contours, see [17]. In the same

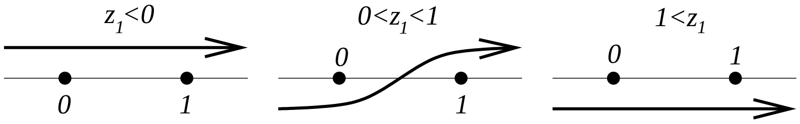


**Figure 1:** The contour of $w_1$ depends on the value of $z_1$.

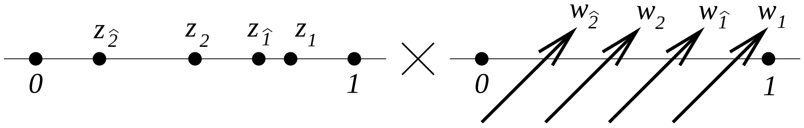


**Figure 2:** Contribution from $\tau = (\hat{2}2\hat{1}1)$.

way, the third line determines how to align contours of $w_i, w_{\hat{i}}$ keeping away from one another. For example, if $z_1 > z_2$, the contour of $w_1$ should be aligned below that of $w_2$. In this way the contours of $(w_i, w_{\hat{i}})$ are aligned differently according to the values of $(z_i, z_{\hat{i}})$. Thus the integral $I$ is expressed as a sum over different orders of magnitude of $(z_i, z_{\hat{i}})$. Labeling different orders of $z$'s by $\tau$, it is expressed as

$$I = \sum_{\tau} J(C_\tau, \alpha, \beta) \times J(P_\tau, \bar{\alpha}, \bar{\beta}) \,. \tag{3.6}$$

Here $C_\tau, P_\tau$ specify the domain of $z$-integration and contours of $w$'s. For example, for $\tau = (\hat{2}2\hat{1}1)$ we have to integrate over the domain

$$0 \le z_{\hat{2}} \le z_2 \le z_{\hat{1}} \le z_1 \le 1 \,,$$

and the contours of $w$'s are arranged so that the contour of $w_{\hat{2}}$ lies on the top, and that of $w_1$ on the bottom, as in the figure 2. The reason why $z$'s are bounded between 0 and 1 is explained shortly.

The integral $J(X_\tau, \alpha, \beta)$ has the following form

$$J(X_\tau, \alpha, \beta) = \int_{X_\tau} d^{m+n} z \prod_{i=1}^{m} z_i^{\alpha} (1-z_i)^{\beta} \prod_{\hat{i}=\hat{1}}^{\hat{n}} z_{\hat{i}}^{\alpha'} (1-z_{\hat{i}})^{\beta'} \left[ \prod z_{ij}^{2\rho} \prod z_{\hat{i}\hat{j}}^{2\rho'} \prod z_{i\hat{j}}^{2\sigma} \right]_\tau . \tag{3.7}$$

The last factor in the integrand is the $\tau$-dependent product of differences, which is positive when the order of $z$'s is precisely as dictated by $\tau$. An example is the following:

$$\left[ \prod z_{ij}^{2\rho} \prod z_{\hat{i}\hat{j}}^{2\rho'} \prod z_{i\hat{j}}^{2\sigma} \right]_{(\hat{2}2\hat{1}1)} = z_{12}^{2\rho} z_{\hat{1}\hat{2}}^{2\rho'} (z_{1\hat{1}} z_{1\hat{2}} z_{\hat{1}2} z_{2\hat{2}})^{2\sigma} \,. \tag{3.8}$$

For skeptical readers, we would like to note that the above definition of $J$ precisely accounts for the phases of the integrand of $I$ in different regions in the complex $z, w$-space. Looking at (3.3) closely, we see that the integrand is positive when the order of $z$'s are the same as that of $w$'s (because we have, for instance, $z_{ij} w_{ij} = x_{ij}^2 - y_{ij}^2 > 0$ in that case) and yields phases when there are relative flips.

Once the arrangement of contours of $w$'s is determined according to the value of $z$'s, the contours can be deformed as long as they do not pass over the points $0, 1$ or cross one another. Then we find that, if one of $z$'s is not in the range $[0, 1]$, then at least one of the contours of $w$'s can slip off to infinity where the integral is assumed to vanish. So we may restrict the value of $z$'s to $[0, 1]$ as was stated in the previous paragraph and implicitly assumed in the figure 2.

The integral $J(P_\tau, \bar{\alpha}, \bar{\beta})$ is given by the integrand of the form (3.7) and the contours of $w_i, w_{\hat{i}}$ arranged according to the order $\tau$. Remarkably, the integral vanishes for certain alignments of contours if we set $\rho = \rho' = 1/2$. To see this, let us consider the following double integral as an example:

$$\int_{-\infty}^{\infty} dw_1 dw_2 (w_1 - w_2)^{2\rho} f(w_1) f(w_2)\,. \tag{3.9}$$

For generic $\rho$ this integral has a phase ambiguity due to the multi-valuedness of $(w_1 - w_2)^{2\rho}$ in the integrand. One has to remove this ambiguity by defining its value to be positive when $w_1 > w_2$ and at the same time deforming the contours of $w_{1,2}$ off the real axis so that they do not cross. However, one does not encounter such problems in the case $\rho = 1/2$ where the integral actually vanishes due to the permutation symmetry. Namely, the contribution from $w_1 > w_2$ and that from $w_1 < w_2$ cancels each other. Returning to the $(w_i, w_{\hat{i}})$-integral $J(P_\tau, \bar{\alpha}, \bar{\beta})$, we thus arrive at the following

**Lemma.** *If the multiple-contour $P_\tau$ contains two contours of $w_i$ and $w_j$ (or those of $w_{\hat{i}}$ and $w_{\hat{j}}$) just next to each other, the integral vanishes. In other words, it is non-vanishing only when the contours of $w_i$ and those of $w_{\hat{i}}$ are arranged alternately.*

This property dramatically reduces the sums over $\tau$ to those of only one or two terms, up to permutations within $w_i$'s or within $w_{\hat{i}}$'s. There are two terms when $m = n$, only one when $m = n \pm 1$. Otherwise the integral itself vanishes. This states that the non-conservation of winding number is up to $\pm 1$ for three-point correlators, which is in accord with the result of FZZ.

Let us summarize here the result up to now. We studied the integral

$$\begin{aligned} I = \frac{1}{m!n!} \int \prod_{i=1}^{m} \left( d^2 z_i z_i^{\alpha} \bar{z}_i^{\bar{\alpha}} (1 - z_i)^{\beta} (1 - \bar{z}_i)^{\bar{\beta}} \right) \prod_{i<j}^{m} |z_{ij}|^{4\rho} \times \\ \times \prod_{\hat{i}=\hat{1}}^{n} \left( d^2 z_{\hat{i}} z_{\hat{i}}^{\alpha'} \bar{z}_{\hat{i}}^{\bar{\alpha}'} (1 - z_{\hat{i}})^{\beta'} (1 - \bar{z}_{\hat{i}})^{\bar{\beta}'} \right) \prod_{\hat{i}<\hat{j}}^{n} |z_{\hat{i}\hat{j}}|^{4\rho'} \prod_{i,\hat{j}}^{m,n} |z_{i\hat{j}}|^{4\sigma}\,, \end{aligned} \tag{3.10}$$

with $\rho = \rho' = 1/2$. This was shown to be non-vanishing only when $m - n = \pm 1$ or $0$. For $m = n + 1$ we have

$$I = (i/2)^{m+n} \int_{C_\tau} d^{n+1} z_i d^n z_{\hat{i}} \prod_{i=1}^{n+1} z_i^{\alpha} (1 - z_i)^{\beta} \prod_{\hat{i}=\hat{1}}^{n} z_{\hat{i}}^{\alpha'} (1 - z_{\hat{i}})^{\beta'} \left[ \prod z_{ij} \prod z_{\hat{i}\hat{j}} \prod z_{i\hat{j}}^{2\sigma} \right]_\tau \times$$

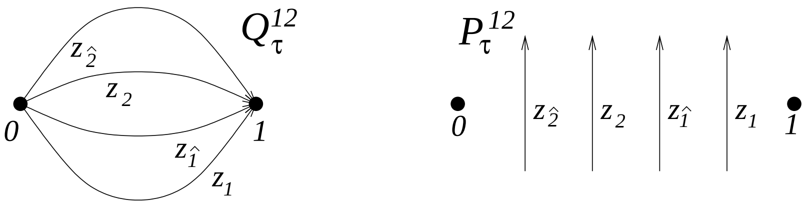


**Figure 3:** Integrals $Q_\tau^{12}$ and $P_\tau^{12}$ for $\tau = (\hat{2}2\hat{1}1)$.

$$\times \int_{P_\tau} d^{n+1}w_i d^n w_{\hat{i}} \prod_{i=1}^{n+1} w_i^{\bar{\alpha}}(1-w_i)^{\bar{\beta}} \prod_{\hat{i}=\hat{1}}^{n} w_{\hat{i}}^{\bar{\alpha}'}(1-w_{\hat{i}})^{\bar{\beta}'} \left[\prod w_{ij} \prod w_{\hat{i}\hat{j}} \prod w_{i\hat{j}}^{2\sigma}\right]_\tau \tag{3.11}$$

with $\tau = ((n+1)\hat{n}n\cdots\hat{2}2\hat{1}1)$. For $m = n$ we have

$$\begin{aligned} I = (i/2)^{m+n} &\int_{C_\tau} d^n z_i d^n z_{\hat{i}} \prod_{i=1}^{n} z_i^{\alpha}(1-z_i)^{\beta} \prod_{\hat{i}=\hat{1}}^{n} z_{\hat{i}}^{\alpha'}(1-z_{\hat{i}})^{\beta'} \left[\prod z_{ij} \prod z_{\hat{i}\hat{j}} \prod z_{i\hat{j}}^{2\sigma}\right]_\tau \times \\ &\times \int_{P_\tau} d^n w_i d^n w_{\hat{i}} \prod_{i=1}^{n} w_i^{\bar{\alpha}}(1-w_i)^{\bar{\beta}} \prod_{\hat{i}=\hat{1}}^{n} w_{\hat{i}}^{\bar{\alpha}'}(1-w_{\hat{i}})^{\bar{\beta}'} \left[\prod w_{ij} \prod w_{\hat{i}\hat{j}} \prod w_{i\hat{j}}^{2\sigma}\right]_\tau + \\ &+ (\tau \to -\tau) \end{aligned} \tag{3.12}$$

with $\tau = (\hat{n}n\cdots\hat{2}2\hat{1}1)$ and $-\tau$ the reversal of $\tau$. The combinatoric factor $(m!n!)^{-1}$ just cancels with the number of equivalent orders of integration variables.

### 3.2 Evaluating the contour integral

We now turn to the evaluation of the integral

$$J(X_\tau) = \int_X d^m z_i d^n z_{\hat{i}} \prod_{i=1}^{m} z_i^{\alpha}(1-z_i)^{\beta} \prod_{\hat{i}=\hat{1}}^{n} z_{\hat{i}}^{\alpha'}(1-z_{\hat{i}})^{\beta'} \left[\prod z_{ij} \prod z_{\hat{i}\hat{j}} \prod z_{i\hat{j}}^{2\sigma}\right]_\tau \tag{3.13}$$

with various contours $X_\tau$ and the order $\tau$. We define the integrand to be single-valued in the region $z_i, \hat{z}_{\hat{i}} \notin [-\infty, 0] \cup [1, \infty]$, and strictly positive when all $z$'s are on $(0, 1)$ with the order dictated by $\tau$. For later convenience, we introduce the parameters $\gamma, \gamma'$ by the followiong relations

$$\alpha + \beta + \gamma = -m - 2\sigma n - 1\,, \qquad \alpha' + \beta' + \gamma' = -2\sigma m - n - 1\,.$$

The parameters $\alpha$, $\alpha'$, $\beta$, $\beta'$, $\gamma$, $\gamma'$ correspond to the quantum numbers of three tachyon vertices.

To begin with, let us define some notation for integrals. Let us define $Q_\tau^{12}$ as the contour integral (3.13) with each contour originating from 0 and heading for 1. The contours are aligned according to $\tau$ and heading upwards. Similarly, define $P_\tau^{12}$ by the integral (3.13) with each contour passing between 0 and 1. Each contour sees 0 on the left and 1 on the right, and the contours are again aligned according to $\tau$. See the figure 3 for a pictorial understanding. As a generalization, we define the integrals $P_\tau^{ab}$ and $Q_\tau^{ab}$ by the interchange of the three vertices. Namely, they correspond to correlators with $a$-th and $b$-th vertices located at the points 0 and 1. Similarly, we define $C_\tau^{12}$ to be the integral $J(C_\tau)$ of the previous subsection, and define $C_\tau^{ab}$ by permutations of vertices. From these definition follows the relations

$$P_\tau^{ab} = P_{-\tau}^{ba}\,, \qquad Q_\tau^{ab} = Q_{-\tau}^{ba}\,, \qquad C_\tau^{ab} = C_{-\tau}^{ba}. \tag{3.14}$$

which hold without the phase ambiguity. One can also find that the contour $P_\tau$ of the previous section corresponds to $P_\tau^{12}$ in the new notation.

Here we would like to note on the complex phases of the above integrals. Writing the contours of $P_\tau^{12}$ as in the right of the figure 3 we find that the complex conjugation serves as the flips of the orientation of contours. Since this yields the factor $(-)^{m+n}$, it follows $P_\tau^{ab}$ is $i^{m+n}$ times a real quantity. $C_\tau^{ab}$ are real by definition.

Each of the above integrals is well-defined and analytic only in a finite region in the parameter space. For example, depending on the value of $\alpha$ and $\alpha'$, the integral $C_\tau^{12}$ may diverge due to the contribution from $z_i, z_{\hat{i}} \sim 0$. By a suitable approximate integration one finds when such divergence occurs. In the case $m = n$ we find that $C_\tau^{12}$ is well-defined when the real parts of

$$1+\alpha+\alpha'-nk'\,, \qquad 1+\beta+\beta'-nk'\,, \qquad 1+\alpha'\,, \qquad 1+\beta$$

are positive, while in the case $m = n+1$ the integral $C_\tau^{12}$ turns out to be well-defined when

$$1+\alpha+\alpha'-nk'\,, \qquad 1+\beta+\beta'-nk'\,, \qquad 1+\alpha\,, \qquad 1+\beta$$

have positive real parts.[1]

Using the integrals defined above, $I$ is expressed as

$$\begin{aligned} (m=n+1) \qquad I &= (i/2)^{m+n} C_\tau^{12}[\alpha,\beta] P_\tau^{12}[\bar\alpha,\bar\beta]\,, \\ (m=n) \qquad I &= (i/2)^{m+n} \left\{ C_\tau^{12}[\alpha,\beta] P_\tau^{12}[\bar\alpha,\bar\beta] + C_{-\tau}^{12}[\alpha,\beta] P_{-\tau}^{12}[\bar\alpha,\bar\beta] \right\}. \end{aligned} \tag{3.15}$$

Thus the problem reduces to the calculation of contour integrals $P_\tau^{ab}, Q_\tau^{ab}$ and $C_\tau^{ab}$. To obtain them, we start with revealing the relations between $Q_\tau^{12}$ and $C_\tau^{12}$. Proceeding recursively we find,

$$\begin{aligned} Q_{(1)}^{12} &= C_{(1)}^{12}\,, \\ Q_{(\hat{1}1)}^{12} &= C_{(\hat{1}1)}^{12} + e^{-2\pi i\sigma} C_{(1\hat{1})}^{12}\,, \\ Q_{(2\hat{1}1)}^{12} &= (1-e^{-4\pi i\sigma}) C_{(2\hat{1}1)}^{12}\,, \\ &\vdots \end{aligned} \tag{3.16}$$

[1] One might expect that the condition must be more strict in the case $m = n+1$, because more contours are ending on the points 0 and 1 as compared to the case $m = n$. Actually, using the techniques of manipulating contours which will be explained in the following, we may replace one of the $2n+1$ contours with the one which encloses all the other contours and has both ends on the same point (0 or 1). So the singularities arise in the same way as in the case $m = n$.

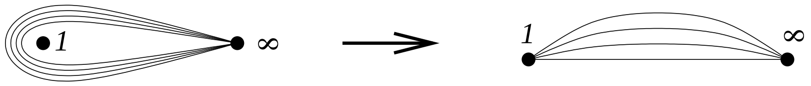


**Figure 4:** Relation between $P^{12}_\tau$ and $Q^{23}_\tau$.

In this way we obtain the following relations:

$$(m=n+1) \qquad Q^{12}_\tau = e^{i\pi n(n+1)k'/2}(-2i)^n \prod_{j=1}^{n} \mathbf{s}(jk')\; C^{12}_\tau\,,$$
$$(m=n) \qquad Q^{12}_\tau = e^{i\pi n(n-1)k'/2}(-2i)^{n-1} \prod_{j=1}^{n-1} \mathbf{s}(jk')\left\{C^{12}_\tau - e^{i\pi nk'}C^{12}_{-\tau}\right\} \tag{3.17}$$

where we used $k' = -2\sigma - 1$ and $\mathbf{s}(x) \equiv \sin(\pi x)$.

We can also find the relation between $P^{ab}_\tau$ and $Q^{ab}_\tau$ using a similar recursion. To do this, deform each of the contours in $P^{ab}_\tau$ so that they all can be seen as starting from $\infty$, encircling around 1 clockwise and ending on $\infty$ as depicted in the left of the figure 4. Dividing each contour into halves, we can relate the contour integral with another one, in which all the contours end on 1 and $\infty$. After suitable change of variables we obtain the relation between $P^{12}_\tau$ and $Q^{23}_\tau$. For the first few cases it becomes

$$\begin{aligned}
\tau = (1) \quad \to \quad P^{12}_\tau &= (e^{-i\pi\beta} - e^{i\pi\beta})Q^{23}_\tau \\
&= -2i\mathbf{s}(\beta)C^{23}_\tau, \\
\tau = (\hat{1}1) \quad \to P^{12}_\tau &= -2i\mathbf{s}(\beta)\left\{e^{-i\pi\beta'}Q^{23}_\tau - e^{i\pi(\beta'+2\sigma)}Q^{23}_{-\tau}\right\} \\
&= (-2i)^2\mathbf{s}(\beta)\left\{\mathbf{s}(\beta')C^{23}_\tau + \mathbf{s}(\beta'+2\sigma)C^{23}_{-\tau}\right\}, \\
\tau = (2\hat{1}1) \quad \to \quad P^{12}_\tau &= -2i\mathbf{s}(\beta)(e^{-i\pi(\beta+\beta')} - e^{i\pi(\beta+\beta'+4\sigma)})Q^{23}_\tau \\
&= (-2i)^3\mathbf{s}(\beta)\mathbf{s}(k')\mathbf{s}(\beta+\beta'-k')C^{23}_\tau, \\
&\vdots
\end{aligned} \tag{3.18}$$

Proceeding in this way we obtain the relations

$$\begin{aligned}
&(m=n+1) \\
P^{12}_\tau &= (-2i)^m e^{-i\pi n(n+1)k'/2}\mathbf{s}(\beta)\prod_{j=1}^{n}\mathbf{s}(\beta+\beta'-jk')Q^{23}_\tau \\
&= (-2i)^{m+n}\mathbf{s}(\beta)\prod_{j=1}^{n}\left[\mathbf{s}(jk')\mathbf{s}(\beta+\beta'-jk')\right]C^{23}_\tau, \\
&(m=n) \\
P^{12}_\tau &= (-2i)^m e^{-i\pi n(n-1)k'/2}\mathbf{s}(\beta)\prod_{j=1}^{n-1}\mathbf{s}(\beta+\beta'-jk')\left\{e^{-i\pi\beta'}Q^{23}_\tau + e^{i\pi(\beta'-nk')}Q^{23}_{-\tau}\right\} \\
&= (-2i)^{m+n}\mathbf{s}(\beta)\prod_{j=1}^{n-1}\left[\mathbf{s}(jk')\mathbf{s}(\beta+\beta'-jk')\right]\left\{\mathbf{s}(\beta')C^{23}_\tau - \mathbf{s}(\beta'-nk')C^{23}_{-\tau}\right\}.
\end{aligned} \tag{3.19}$$

These relations together with the symmetry (3.14) largely determines the functional form of $P_\tau^{ab}, Q_\tau^{ab}$ and $C_\tau^{ab}$. We analyze the cases $m = n$ and $m = n + 1$ separately in the following.

### 3.3 The formula for $m = n + 1$

First we study the simpler case $m = n+1$ where the only non-trivial $\tau$ is symmetric under the reversal, i.e., $\tau = -\tau$. By exchanging the vertices in the previous relations we obtain the relations of the form:

$$(i/2)^{m+n} P_\tau^{12} = C_\tau^{13}\, \mathbf{s}(\alpha) \prod_{j=1}^{n} \mathbf{s}(jk')\mathbf{s}(\alpha+\alpha'-jk') = C_\tau^{23}\, \mathbf{s}(\beta) \prod_{j=1}^{n} \mathbf{s}(jk')\mathbf{s}(\beta+\beta'-jk')\,. \tag{3.20}$$

Using them together with the symmetry (3.14) we can guess, as in [16],

$$\begin{aligned}
C_\tau^{12} &= \lambda_n(\sigma) \frac{\Gamma(1+\alpha)\Gamma(1+\beta)}{\Gamma(-\gamma)} \prod_{j=1}^{n} \left[\frac{\Gamma(1+\alpha+\alpha'-jk')\Gamma(1+\beta+\beta'-jk')}{\Gamma(jk'-\gamma-\gamma')}\right],\\
P_\tau^{12} &= \mu_n(\sigma) \frac{\Gamma(1+\gamma)}{\Gamma(-\alpha)\Gamma(-\beta)} \prod_{j=1}^{n} \left[\frac{\Gamma(1+\gamma+\gamma'-jk')}{\Gamma(jk'-\alpha-\alpha')\Gamma(jk'-\beta-\beta')}\right],\\
(i/2)^{m+n}\mu_n(\sigma) &= -\pi^{n+1}\lambda_n(\sigma) \prod_{j=1}^{n} \mathbf{s}(-jk').
\end{aligned} \tag{3.21}$$

The above form is also expected from the analytic structure of the integrals. In the previous subsection we discussed the region in the parameter space where the specific integrals are well-defined. In the same way we can study where in the parameter space the integrals have singularities. Taking the alignments of contours into account, we can convince ourselves that the singularities of the integrals in the parameter space are precisely as expressed in the above ansatz. It follows that $\lambda_n$ and $\mu_n$ have no poles in the parameter space, so they are independent of $\alpha, \alpha', \beta, \beta'$. Although our argument is not rigorous, we could in principle follow the same steps as in [16] to introduce many "intermediate" contour integrals that connect between $P_\tau^{ab}$'s and $C_\tau^{ab}$'s and do the analytic continuation strictly. In the following we would rather give some evidences supporting the above ansatz.

We can find the asymptotic behavior of the expression at, say, large $\alpha, \alpha'$ with their ratio fixed. The integral $C_\tau^{12}$ behaves then like

$$\begin{aligned}
C_\tau^{12} &= \int d^m z_i d^n z_{\hat{i}} \prod_{i=1}^{m} z_i^{\alpha}(1-z_i)^{\beta} \prod_{\hat{i}=\hat{1}}^{n} z_{\hat{i}}^{\alpha'}(1-z_{\hat{i}})^{\beta'} \left[\prod z_{ij} \prod z_{\hat{i}\hat{j}} \prod z_{i\hat{j}}^{2\sigma}\right]_\tau \\
&= \alpha^{-m-n} \int d^m\xi_i d^n\xi_{\hat{i}}\; e^{-(\Sigma_i\xi_i+\Sigma_{\hat{i}}\xi_{\hat{i}})/\alpha} \prod_{i=1}^{m} e^{-\xi_i}(1-e^{-\xi_i/\alpha})^{\beta} \prod_{\hat{i}=\hat{1}}^{n} e^{-\alpha'\xi_{\hat{i}}/\alpha}(1-e^{-\xi_{\hat{i}}/\alpha})^{\beta'} \times \\
&\quad \times \left[\prod(e^{-\xi_i/\alpha}-e^{-\xi_j/\alpha}) \prod(e^{-\xi_{\hat{i}}/\alpha}-e^{-\xi_{\hat{j}}/\alpha}) \prod(e^{-\xi_i/\alpha}-e^{-\xi_{\hat{j}}/\alpha})^{2\sigma}\right]_\tau \\
&\sim \alpha^{-2n-1-n(\beta+\beta')-\beta-n^2-2n(n+1)\sigma}(1+\mathcal{O}(\alpha^{-1}))\,.
\end{aligned}$$


**Figure 5:** The contours for evaluating the limit $\alpha \to -1$.

On the other hand, using the asymptotic behavior of $\Gamma$ function

$$\Gamma(x) \sim \sqrt{2\pi} e^{-x} x^{x-1/2} \tag{3.22}$$

we find the asymptotic behavior of $C_\tau^{12}$ guessed in (3.21) at large $\alpha, \alpha'$ is

$$C_\tau^{12} \sim \alpha^{k'n(n+1)-\beta-1-n(1+\beta+\beta')}$$

in precise agreement with the above.

Another useful information on the integrals $P_\tau^{ab}$ is the relation between $P_\tau^{ab}|_{\alpha\to-1}$ and the one with fewer contours, from which we can determine the functional form of $\mu(\sigma)$. Let us consider $P_\tau^{12}$ in the limit $\alpha \to -1$, assuming that the leftmost variable in $\tau$ is without the hat. The limit can be evaluated by encircling the leftmost contour around 0 and the rest around 1, as in the figure 5.

In the limit $\alpha \to -1$ the integral over the leftmost contour is well approximated by the contribution near 0. At $\alpha = -1$ the leftmost contour can then be replaced by the one enclosing the single pole at $z = 0$, and the integral is simply given by $2\pi i$ times the residue. Thus we have a relation between $P^{12}[m, n; \alpha, \alpha', \beta, \beta']$:

$$P_\tau^{12}[m, n; -1, \alpha', \beta, \beta'] = 2\pi i P_\tau^{12}[m-1, n; 0, \alpha' + 2\sigma, \beta, \beta']\,. \tag{3.23}$$

Applying this argument twice we find

$$P_\tau^{12}[m, n; -1, -1-2\sigma, \beta, \beta'] = -4\pi^2 P_\tau^{12}[m-1, n-1; 2\sigma, 0, \beta, \beta']\,. \tag{3.24}$$

This gives the recursion relation for $\mu_n(\sigma)$,

$$\mu_n = \frac{-4\pi^2 \mu_{n-1}(\sigma)}{\Gamma(1+nk')}\,. \tag{3.25}$$

Combined with $\lambda_0 = 1, \mu_0 = 2\pi i$ we obtain

$$\mu_n = \frac{(2\pi i)^{2n+1}}{\prod_{j=1}^n \Gamma(1+jk')}\,, \qquad \lambda_n = \prod_{j=1}^n \Gamma(-jk')\,. \tag{3.26}$$

In all, the formula for $I$ in the case $m = n+1$ becomes:

$$I = -\pi^{m+n}(-)^{\gamma-\bar\gamma} \frac{\Gamma(1+\alpha)\Gamma(1+\beta)\Gamma(1+\gamma)}{\Gamma(-\bar\alpha)\Gamma(-\bar\beta)\Gamma(-\bar\gamma)} \times \tag{3.27}$$
$$\times \prod_{j=1}^n \frac{\Gamma(-jk')\Gamma(1+\alpha+\alpha'-jk')\Gamma(1+\beta+\beta'-jk')\Gamma(1+\gamma+\gamma'-jk')}{\Gamma(1+jk')\Gamma(jk'-\alpha-\alpha')\Gamma(jk'-\beta-\beta')\Gamma(jk'-\gamma-\gamma')}\,.$$

Rewriting $\alpha, \beta, \gamma$ in terms of the quantum numbers of coset CFT via (2.15) and introducing the notation $\Delta(x) = \Gamma(x)/\Gamma(1-x)$, we obtain the following:

$$I = -(-)^{m_3+\bar{m}_3}\pi^{m+n}\frac{\Gamma(m_1+j_1+1)\Gamma(m_2+j_2+1)\Gamma(m_3+j_3+1)}{\Gamma(\bar{m}_1-j_1)\Gamma(\bar{m}_2-j_2)\Gamma(\bar{m}_3-j_3)}\times$$
$$\times\left[\prod_{s=1}^{n}\Delta(1+sk')\Delta(sk'-2j_1)\Delta(sk'-2j_2)\Delta(sk'-2j_3)\right]^{-1}. \tag{3.28}$$

We further rewrite this using the $\Upsilon$ function introduced in [25] and [26]. This is defined by

$$\ln\Upsilon(x) = \int_0^\infty \frac{dt}{t}\left[\left(\frac{Q}{2}-x\right)^2 e^{-t} - \frac{\sinh^2[(\frac{Q}{2}-x)\frac{t}{2}]}{\sinh\frac{bt}{2}\sinh\frac{t}{2b}}\right], \qquad Q = b + b^{-1}. \tag{3.29}$$

We set the parameter $b$ as $b^{-2} = k'$. $\Upsilon(x)$ has zeroes at

$$x = -mb - nb^{-1},\ Q + mb + nb^{-1}, \qquad (m, n \in \mathbb{Z}_{\geq 0}).$$

Some useful properties are

$$\begin{aligned}
\Upsilon(x+b) &= \Delta(bx)b^{1-2bx}\Upsilon(x),\\
\Upsilon(x+b^{-1}) &= \Delta(x/b)b^{2x/b-1}\Upsilon(x),\\
\Upsilon(Q-x) &= \Upsilon(x),\\
\Upsilon'(-mb^{-1}) &= (-)^m b^{-2m}\Upsilon[(m+1)b^{-1}]\Gamma(m+1)^2.
\end{aligned} \tag{3.30}$$

Using this we can rewrite the above expression as

$$I = (-)^{n+1+m_3+\bar{m}_3}\frac{b^{2nk}\pi^{2n+1}\Gamma(m_1+j_1+1)\Gamma(m_2+j_2+1)\Gamma(m_3+j_3+1)}{\Gamma[b^2(\Sigma j_a+1)+\frac{1}{2}]^2\Gamma(\bar{m}_1-j_1)\Gamma(\bar{m}_2-j_2)\Gamma(\bar{m}_3-j_3)}\times$$
$$\times\frac{\Upsilon[b^{-1}]}{\Upsilon[(\Sigma j_a+1)b+\frac{1}{2b}]}\frac{\prod_{a=1}^{3}\Upsilon[(2j_a+1)b]}{\prod_{a<b}^{3}\Upsilon[-j_{ab}b+\frac{1}{2b}]}. \tag{3.31}$$

where $n = b^2(\sum j_a + 1) - 1/2$ and $j_{ab}$ are defined by $j_{12} = j_1 + j_2 - j_3$, etc.

### 3.4 The formula for $m = n$

The case $m = n$ is a little more complicated since there are essentially two non-equivalent alignments of contours, $\tau$ and $-\tau$. However, one can read off a lot from the relations (3.19) between $P_\tau^{ab}$ and $C_\tau^{ab}$ as in the case $m = n+1$. The relations are simplified if we rescale $C_\tau^{ab}$ and $P_\tau^{ab}$ as

$$C_\tau^{12} = \lambda_n c_\tau^{12}\cdot\prod_{j=1}^{n-1}\left[\frac{\Gamma(1+\alpha+\alpha'-jk')\Gamma(1+\beta+\beta'-jk')}{\Gamma(jk'-\gamma-\gamma')}\right], \tag{3.32}$$

$$(i/2)^{m+n}P_\tau^{12} = \pi^{n-1}\lambda_n p_\tau^{12}\cdot\prod_{j=1}^{n-1}\left[\frac{\mathbf{s}(-jk')\Gamma(1+\gamma+\gamma'-jk')}{\Gamma(jk'-\alpha-\alpha')\Gamma(jk'-\beta-\beta')}\right]. \tag{3.33}$$

Using them we can rewrite (3.19) neatly in a matrix form:

$$\begin{bmatrix} p_\tau^{12} \\ p_\tau^{21} \end{bmatrix} = A_\beta \begin{bmatrix} c_\tau^{23} \\ c_\tau^{32} \end{bmatrix} = A_\alpha^T \begin{bmatrix} c_\tau^{31} \\ c_\tau^{13} \end{bmatrix}, \qquad \begin{bmatrix} p_\tau^{23} \\ p_\tau^{32} \end{bmatrix} = A_\gamma \begin{bmatrix} c_\tau^{31} \\ c_\tau^{13} \end{bmatrix} = A_\beta^T \begin{bmatrix} c_\tau^{12} \\ c_\tau^{21} \end{bmatrix},$$
$$\begin{bmatrix} p_\tau^{31} \\ p_\tau^{13} \end{bmatrix} = A_\alpha \begin{bmatrix} c_\tau^{12} \\ c_\tau^{21} \end{bmatrix} = A_\gamma^T \begin{bmatrix} c_\tau^{23} \\ c_\tau^{32} \end{bmatrix}, \tag{3.34}$$

where

$$A_\alpha = \begin{bmatrix} \mathbf{s}(\alpha)\mathbf{s}(\alpha') & -\mathbf{s}(\alpha)\mathbf{s}(\alpha'-nk') \\ -\mathbf{s}(\alpha')\mathbf{s}(\alpha-nk') & \mathbf{s}(\alpha)\mathbf{s}(\alpha') \end{bmatrix} \tag{3.35}$$

and $A_\beta, A_\gamma$ are defined similarly.

**The case $m = n = 1$.** In proceeding further, let us analyze first the case $m = n = 1$. As we will see, in this case we can perform the contour-integration rather explicitly. The result will be used to write down the formula for more generic cases. The integral $C_\tau^{12}$ is written in this case as

$$C_\tau^{12} = \int_0^1 dz \int_0^z d\hat{z} z^\alpha (1-z)^\beta \hat{z}^{\alpha'} (1-\hat{z})^{\beta'} (z-\hat{z})^{2\sigma}. \tag{3.36}$$

Using the formulae involving (generalized) hypergeometric functions

$$\int_0^1 dt t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} F(a,b;c;z), \tag{3.37}$$
$$\int_0^1 dt t^{c-1}(1-t)^{f-c-1} F(a,b;e;t) = \frac{\Gamma(c)\Gamma(f-c)}{\Gamma(f)} {}_3F_2(a,b,c;e,f;1), \tag{3.38}$$

we can carry out the integration and obtain

$$\begin{aligned} C_\tau^{12} &= \frac{\Gamma(\alpha+\alpha'-k'+1)\Gamma(\beta+1)\Gamma(\alpha'+1)\Gamma(-k')}{\Gamma(\alpha+\alpha'+\beta-k'+2)\Gamma(\alpha'-k'+1)} \times \\ &\quad \times {}_3F_2\Big(-\beta', \alpha'+1, \alpha+\alpha'-k'+1; \alpha'-k'+1, \alpha+\alpha'+\beta-k'+2; 1\Big) \\ &= \frac{\Gamma(\alpha+\alpha'-k'+1)\Gamma(\beta+\beta'-k'+1)\Gamma(\alpha'+1)\Gamma(\beta+1)}{\Gamma(\alpha'-\gamma+1)\Gamma(\beta-\gamma'+1)} \times \\ &\quad \times {}_3F_2\Big(\alpha'+1, \beta+1, k'-\gamma-\gamma'; \alpha'-\gamma+1, \beta-\gamma'+1; 1\Big). \end{aligned} \tag{3.39}$$

In the above we repeatedly used the formula

$${}_3F_2(a,b,c;e,f;1) = \frac{\Gamma(e)\Gamma(f)\Gamma(s)}{\Gamma(a)\Gamma(s+b)\Gamma(s+c)} {}_3F_2(e-a, f-a, s; s+b, s+c; 1) \tag{3.40}$$

with $s = e+f-a-b-c$, and the relations between parameters $\alpha+\beta+\gamma+1 = \alpha'+\beta'+\gamma'+1 = k'$. The next formula we use is the relation between $G\begin{bmatrix} a,b,c \\ e,f \end{bmatrix} \equiv$

$\frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(e)\Gamma(f)}{}_3F_2(a,b,c;e,f;1)$:

$$G\begin{bmatrix} a,b,c \\ e,f \end{bmatrix} = \frac{\mathbf{s}(e-b)\mathbf{s}(f-b)}{\mathbf{s}(a)\mathbf{s}(c-b)} G\begin{bmatrix} b, 1+b-e, 1+b-f \\ 1+b-c, 1+b-a \end{bmatrix} + \frac{\mathbf{s}(e-c)\mathbf{s}(f-c)}{\mathbf{s}(a)\mathbf{s}(b-c)} G\begin{bmatrix} c, 1+c-e, 1+c-f \\ 1+c-b, 1+c-a \end{bmatrix}, \tag{3.41}$$

from which we can derive the relations between $C_\tau^{ab}$ of the form:

$$\mathbf{s}(\alpha'-\beta)C_\tau^{12} = \mathbf{s}(\gamma)C_\tau^{13} - \mathbf{s}(\gamma')C_\tau^{32}. \tag{3.42}$$

Looking closely, the set of relations of this form turn to be equivalent to (3.19) or (3.34). Summarizing, in the case $m=n=1$ the integral $I$ is given by

$$I = \int d^2z d^2w z^\alpha(1-z)^\beta \bar{z}^{\bar{\alpha}}(1-\bar{z})^{\bar{\beta}} w^{\alpha'}(1-w)^{\beta'} \bar{w}^{\bar{\alpha}'}(1-\bar{w})^{\bar{\beta}'}|z-w|^{4\sigma}. \tag{3.43}$$

It is decomposed into contour integrals

$$I = (i/2)^2\left\{C_\tau^{12}[\alpha,\beta]P_\tau^{12}[\bar{\alpha},\bar{\beta}] + C_{-\tau}^{12}[\alpha,\beta]P_{-\tau}^{12}[\bar{\alpha},\bar{\beta}]\right\}. \tag{3.44}$$

The integrals $C_\tau^{ab}$ are given by the formula

$$C_\tau^{12} = \frac{\Gamma(1+\alpha+\alpha'-k')\Gamma(1+\beta+\beta'-k')}{\Gamma(k'-\gamma-\gamma')} G\begin{bmatrix} \alpha'+1, \beta+1, k'-\gamma-\gamma' \\ \alpha'-\gamma+1, \beta-\gamma'+1 \end{bmatrix}, \tag{3.45}$$

and $P_\tau^{ab}$ are related to $C_\tau^{ab}$ via

$$(i/2)^2\begin{bmatrix} P_\tau^{12} \\ P_\tau^{21} \end{bmatrix} = A_\beta \begin{bmatrix} C_\tau^{23} \\ C_\tau^{32} \end{bmatrix} = A_\alpha^T \begin{bmatrix} C_\tau^{31} \\ C_\tau^{13} \end{bmatrix}, \quad A_\alpha = \begin{bmatrix} \mathbf{s}(\alpha)\mathbf{s}(\alpha') & -\mathbf{s}(\alpha)\mathbf{s}(\alpha'-k') \\ -\mathbf{s}(\alpha')\mathbf{s}(\alpha-k') & \mathbf{s}(\alpha)\mathbf{s}(\alpha') \end{bmatrix}. \tag{3.46}$$

**Generic cases with $m = n$.** Although it is extremely difficult to carry out the integration explicitly in generic cases, we find that the relations (3.34) among $c_\tau^{ab}$ and $p_\tau^{ab}$ have the same form as in the case $m=n=1$. If we assume $c_\tau^{12}$ to be given by replacing $k'$ with $nk'$ in (3.45), we have

$$C_\tau^{12} = \lambda_n \prod_{j=1}^{n}\left[\frac{\Gamma(1+\alpha+\alpha'-jk')\Gamma(1+\beta+\beta'-jk')}{\Gamma(jk'-\gamma-\gamma')}\right] \times \times G\begin{bmatrix} \alpha'+1, \beta+1, nk'-\gamma-\gamma' \\ \alpha'-\gamma+1, \beta-\gamma'+1 \end{bmatrix} \tag{3.47}$$

with $\lambda_1 = 1$. From the recursion relation (3.23) we obtain $\lambda_n$:

$$\lambda_n = \prod_{j=1}^{n-1}\Gamma(-jk'). \tag{3.48}$$

Putting all these together, we arrive at the following expression for $I$

$$I = \pi^{2n-2}\prod_{j=1}^{n-1}\left[\frac{\Gamma(-jk')\Gamma(1+\alpha+\alpha'-jk')\Gamma(1+\beta+\beta'-jk')\Gamma(1+\gamma+\gamma'-jk')}{\Gamma(1+jk')\Gamma(jk'-\alpha-\alpha')\Gamma(jk'-\beta-\beta')\Gamma(jk'-\gamma-\gamma')}\right]\times$$
$$\times\int d^2zd^2wz^{\alpha}(1-z)^{\beta}\bar{z}^{\bar{\alpha}}(1-\bar{z})^{\bar{\beta}}w^{\alpha'}(1-w)^{\beta'}\bar{w}^{\bar{\alpha}'}(1-\bar{w})^{\bar{\beta}'}|z-w|^{-2-2nk'}\,. \quad (3.49)$$

Rewriting it further using $\Upsilon$ function and $b = k'^{-1/2}$ we obtain

$$I = I_1(j_a, m_a, \bar{m}_a)\frac{(i\pi b^2)^{2n-2}}{\Gamma[b^2(\Sigma j_a+1)]^2}\frac{\Upsilon[b]\prod_{a=1}^3\Upsilon[(2j_a+1)b]}{\Upsilon[(\Sigma j_a+1)b]\prod_{a<b}^3\Upsilon[(j_{ab}+1)b]}\,, \quad (3.50)$$

$$I_1(j_a, m_a, \bar{m}_a) = \int d^2zd^2wz^{j_1+m_1}(1-z)^{j_2+m_2}\bar{z}^{j_1-\bar{m}_1}(1-\bar{z})^{j_2-\bar{m}_2}\times$$
$$\times w^{j_1-m_1}(1-w)^{j_2-m_2}\bar{w}^{j_1+\bar{m}_1}(1-\bar{w})^{j_2+\bar{m}_2}|z-w|^{-2-2nk'}\,. \quad (3.51)$$

## 4. A check of the duality

As was mentioned in the introduction, the sine-Liouville theory is believed to be dual to the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ coset model. We check this duality by comparing the three-point functions of primary operators in the two theories.

The three-point function of primary fields in the coset model is obtained from the correlation functions of primary fields $\Phi_j(z,\bar{z};x,\bar{x})$ in $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ WZW model. $\Phi_j$ is characterized by the property

$$\Phi_{jm\bar{m}}(z,\bar{z}) = \int d^2x x^{j+m}\bar{x}^{j+\bar{m}}\Phi_j(z,\bar{z};x,\bar{x})\,, \quad (4.1)$$

where $\Phi_{jm\bar{m}}$ are primary fields of $\mathrm{SL}(2,\mathbb{R})$-spin $j$ and $(J^3,\bar{J}^3) = (m,\bar{m})$. As shown in [19] and also in [20], the three-point function of $\Phi_j$ is given by

$$\left\langle\prod_{a=1}^3\Phi_{j_a}(z_a;x_a)\right\rangle = D(j_a)\prod_{a<b}^3|z_{ab}|^{-2h_{ab}}|x_{ab}|^{-2j_{ab}-2}\,,$$
$$D(j_a) = \frac{b^2\pi}{2}\frac{[k^{-1}b^{-2b^2}\Delta(b^2)]^{-\Sigma j_a-2}\Upsilon[b]\Upsilon[(2j_1+2)b]\Upsilon[(2j_2+2)b]\Upsilon[(2j_3+2)b]}{\Upsilon[(\Sigma j_a+2)b]\Upsilon[(j_{12}+1)b]\Upsilon[(j_{23}+1)b]\Upsilon[(j_{13}+1)b]}\,. \quad (4.2)$$

Comparing this with (3.50), we find that the part that depends only on $j_a$ is in a good agreement.[2] As for the part that depends also on $m_a$ and $\bar{m}_a$, the three-point

[2] Actually the agreement is almost precise if we take into account the effect of wave-function renormalization of $\Phi_j$. Note also that the three-point function in sine-Liouville theory has an additional factor $\sim\Gamma(-2n)\Gamma(1+2n)$ which arises from the integration over the $\rho$-zeromode and the expansion of $(S^+_{\mathrm{int}}+S^-_{\mathrm{int}})^{2n}$.



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function in coset model has the following factor

$$\int \prod_{a=1}^{3} [d^2 x_a x_a^{j_a+m_a} \bar{x}_a^{j_a-\bar{m}_a}] \prod_{a<b}^{3} |x_{ab}|^{-2j_{ab}-2} \sim \delta\left(\sum m_a\right) \delta\left(\sum \bar{m}_a\right) F(j_a, m_a, \bar{m}_a)\,, \tag{4.3}$$

where the function $F$ is the left-right combined version of the $3j$-symbol:

$$\begin{aligned} F(j_a, m_a, \bar{m}_a) \equiv \int & d^2x_1 d^2x_2 x_1^{j_1+m_1} \bar{x}_1^{j_1-\bar{m}_1} x_2^{j_2+m_2} \bar{x}_2^{j_2-\bar{m}_2} \times \\ & \times |1-x_1|^{-2j_{13}-2} |1-x_2|^{-2j_{23}-2} |x_{12}|^{-2j_{12}-2}\,. \end{aligned} \tag{4.4}$$

which can also be written in terms of generalized hypergeometric functions. The explicit form is found in [21]. So it represents the group-theoretical structure of the correlation functions. Quite surprisingly, using the formulae involving generalized hypergeometric functions we can show that $F$ precisely coincides with $I_1$ given in (3.51). Hence we conclude that, in the case where the winding number conserves, the three-point functions in sine-Liouville theory agree with those in the coset model, including the SL(2) group-theoretical coefficients.

## 5. An application

Here we would like to apply our new formula to a problem in superstring theory on a singular Calabi-Yau(CY) manifold. Suppose we compactify the superstring theory on a CY manifold $X$ with a singularity. According to the holographic duality of [4, 5], in the decoupling limit $g_s \to 0$ we end up with a non-gravitational but non-trivial interacting theory on the remaining non-compact space. As a typical case, if we take as $X$ an open non-compact CY two-fold with $A_k$ type singularity, the resultant theory in the decoupling limit is believed to describe the non-trivial field theory on $k$ coincident NS5-branes [22]. Generically the holographic duality states the correspondence between

$$\begin{aligned} &\text{decoupling limit of superstring on } \mathbb{R}^d \times X_n(F=0) \\ &(= \text{a non-trivial field theory on } \mathbb{R}^d) \\ &\Longleftrightarrow \text{superstring on } \mathbb{R}^d \times \mathbb{R}_\rho \times S^1_Y \times LG(W=F). \end{aligned}$$

Here $X_n$ is a CY $n$-fold ($2n+d=10$) defined by the complex equation $F(X)=0$, and $LG$ denotes the corresponding Landau-Ginzburg model with the superpotential $W=F$. Using this correspondence we can study the non-trivial field theories from the dual pictures. Here we shall calculate the three-point amplitudes of certain tachyonic operators using our new formula. The two-point functions can also be calculated in a similar way [23].

The worldsheet theory of a superstring on $\mathbb{R}^d\times\mathbb{R}_\rho\times S^1_Y\times LG$ is given by the action

$$S = S_{\mathbb{R}^d} + S_L + S_{LG}\,, \tag{5.1}$$

where $S_{\mathbb{R}^d}, S_{LG}$ are the actions for $\mathbb{R}^d$ and Landau-Ginzburg degrees of freedom, respectively. The remaining degrees of freedom corresponding to the coordinates $\rho, Y$ are described by the $N=2$ Liouville action

$$\begin{aligned}
S_L &= \frac{1}{2\pi}\int d^2z\bigg(\partial\rho\bar\partial\rho + \partial Y\bar\partial Y - \frac{Q}{4}\rho\sqrt{g}R + \\
&\qquad + \psi_\rho\bar\partial\psi_\rho + \psi_Y\bar\partial\psi_Y + \bar\psi_\rho\partial\bar\psi_\rho + \bar\psi_Y\partial\bar\psi_Y\bigg) + S^+_{\text{int}} + S^-_{\text{int}}\,, \\
S^\pm_{\text{int}} &= \frac{\mu_\pm}{2\pi}\int d^2z(\psi_\rho \pm i\psi_Y)(\bar\psi_\rho \pm i\bar\psi_Y)e^{-\frac{1}{Q}(\rho\pm iY)}\,,
\end{aligned} \tag{5.2}$$

where $Q\equiv\sqrt{2/k}$. The theory reduces to the free theory if the cosmological constant $\mu_\pm$ vanishes. In this case the $N=2$ superconformal algebra is generated by the currents

$$\begin{aligned}
T &= -\frac{1}{2}(\partial\rho\partial\rho + Q\,\partial^2\rho + \partial Y\partial Y + \psi_\rho\partial\psi_\rho + \psi_Y\partial\psi_Y)\,, \\
T_F^\pm &= \frac{i}{2}(\psi_\rho \pm i\psi_Y)\partial(\rho\mp iY) + \frac{iQ}{2}\partial(\psi_\rho\pm i\psi_Y)\,, \\
J &= -i\psi_\rho\psi_Y + iQ\partial Y\,.
\end{aligned} \tag{5.3}$$

Note that the interaction $S^\pm_{\text{int}}$ is made from the screening currents which have no singular OPEs with $N=2$ superconformal currents up to total derivatives.

In the following we calculate the correlators of "tachyons", which are expressed in the $(-1,-1)$-picture as

$$T(z,\bar z) = \exp\left[-\varphi + ik_\mu X^\mu + Q(j\rho - imY_L + i\bar m Y_R)\right], \tag{5.4}$$

where $\varphi$ constitutes the system of bosonized super-ghost together with $\xi$ and $\eta$. We may regard $j$ and $m$ as the quantum numbers of the corresponding operators in the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ Kazama-Suzuki model [24]. The minus sign in front of $m$ is due to the fact that the Liouville interaction terms explicitly break the conservation of $Y$-momentum, which we want to relate to the winding number along the cigar. So the momentum $p$ and the winding number $w$ along the $S^1$ of the cigar are given by

$$m = \frac{1}{2}(p+kw)\,, \qquad \bar m = \frac{1}{2}(p-kw)\,. \tag{5.5}$$

The on-shell condition requires the parameters to satisfy

$$\frac{k_\mu k^\mu}{2} + \frac{m^2 - j(j+1)}{k} = \frac{k_\mu k^\mu}{2} + \frac{\bar m^2 - j(j+1)}{k} = \frac{1}{2}\,. \tag{5.6}$$

The GSO condition requires the physical vertices to have odd worldsheet fermion number. Since it is defined by the U(1) charge of the $N=2$ superconformal symmetry we have,

$$m, \bar{m} \in \frac{k}{2}(2\mathbb{Z}+1). \tag{5.7}$$

The corresponding vertices in the $(0,0)$-picture become

$$\tilde{T}(z,\bar{z}) = [ik_\mu \psi_X^\mu + Qj\psi_\rho - iQm\psi_Y][ik_\mu \bar{\psi}_X^\mu + Qj\bar{\psi}_\rho + iQ\bar{m}\bar{\psi}_Y]e^{ik_\mu X^\mu + Q(j\rho - im_i Y_L + i\bar{m}_i Y_R)}. \tag{5.8}$$

The three-point function of tachyons is defined by

$$\begin{aligned} A &\equiv \left\langle c\bar{c}T_1(z_1)\, c\bar{c}T_2(z_2)\, c\bar{c}\tilde{T}_3(z_3)\right\rangle \\ &= \int [D\rho DY D\psi_\rho D\psi_Y \cdots]\ J_3 \prod_{i=1}^{3} c\bar{c}e^{ik_i X + Q(j_i\rho - im_i Y_L + i\bar{m}_i Y_R)}(z_i)\, e^{-S}\,, \end{aligned} \tag{5.9}$$

$$\begin{aligned} J_3 &= e^{-\varphi(z_1,\bar{z}_1) - \varphi(z_2,\bar{z}_2)} \\ &\quad \times [ik_3\cdot\psi_X + Qj_3\psi_\rho - iQm_3\psi_Y](z_3)[ik_3\cdot\bar{\psi}_X + Qj_3\bar{\psi}_\rho + iQ\bar{m}_3\bar{\psi}_Y](\bar{z}_3)\,. \end{aligned} \tag{5.10}$$

Since the tachyons do not contain the fields in the LG part, we can integrate over them and simply ignore the LG part. As was discussed in [27] (see also [28]), by integrating over the zero-mode of $\rho$ the correlator reduces to that of a free theory with a number of insertions of $S_{\rm int} \equiv S^+_{\rm int} + S^-_{\rm int}$:

$$\begin{aligned} A &\sim Q\Gamma(-s) \int [D\rho DY D\psi_\rho D\psi_Y \cdots]\ J_3 \prod_{i=1}^{3} c\bar{c}e^{ik_i X + Q(j_i\rho - im_i Y_L + i\bar{m}_i Y_R)}(z_i) \times \\ &\qquad\qquad \times (S_{\rm int})^s \exp\left[-S_{\mathbb{R}^d} - S_{L\ (\mu_\pm = 0)}\right], \\ s &= \frac{2}{k}(\Sigma j_i + 1)\,. \end{aligned} \tag{5.11}$$

From the expression of $S_{\rm int}$ we find that only a part of $J_3$ containing $\psi_\rho, \psi_Y$ yields non-zero contribution to $A$. So if we bosonize $\psi_\rho, \psi_Y$ as

$$\psi_\rho \pm i\psi_Y = \sqrt{2}e^{\pm iH_L}\,, \qquad \bar{\psi}_\rho \pm i\bar{\psi}_Y = \sqrt{2}e^{\pm iH_R}\,, \tag{5.12}$$

we may replace the above $J_3$ with

$$J_3 = \frac{1}{k} e^{-\varphi(z_1,\bar{z}_1) - \varphi(z_2,\bar{z}_2)} \left[(j_3 - m_3)(j_3 + \bar{m}_3)e^{iH} + (j_3 + m_3)(j_3 - \bar{m}_3)e^{-iH}\right](z_3,\bar{z}_3)\,.$$

The three-point function is then written as a sum of two terms, $A \sim A_+ + A_-$, with

$$\begin{aligned} A_\pm &= \delta^n\Big(\sum_i k_i\Big)\delta\Big(\sum_i p_i\Big)\delta\Big(\sum_i w_i \mp 1\Big)\frac{(j_3 \mp m_3)(j_3 \pm \bar{m}_3)}{k}\frac{\Gamma(-s)s!}{(\frac{s+1}{2})!(\frac{s-1}{2})!} \times \\ &\quad \times \left\langle e^{-\varphi(z_1) - \varphi(z_2) \pm iH(z_3)} \prod_{i=1}^{3} c\bar{c}e^{ik_i X + Q(j_i\rho - im_i Y_L + i\bar{m}_i Y_R)}(z_i)(S^+_{\rm int})^{\frac{s\mp1}{2}}(S^-_{\rm int})^{\frac{s\pm1}{2}}\right\rangle_{\rm free}, \end{aligned} \tag{5.13}$$

$$S^\pm_{\rm int} = \frac{\mu_\pm}{\pi}\int d^2z e^{\pm iH - \frac{1}{Q}(\rho \pm iY)}\,. \tag{5.14}$$

It is non-zero only when the total winding number is $\pm 1$. So, in this case, the restriction on the total winding number simply comes from the argument of the worldsheet fermion number. This fact was also pointed out in [7].

Let us consider the case $\sum w_i = 1$. The three-point function becomes

$$A_+ = \delta^n\Big(\sum_i k_i\Big)\delta\Big(\sum p_i\Big)\delta\Big(\sum w_i - 1\Big)\frac{\pi(j_3-m_3)(j_3+\bar{m}_3)}{k\mathbf{s}(2n+1)n!(n+1)!}\Big(\frac{\mu_+}{\pi}\Big)^n\Big(\frac{\mu_-}{\pi}\Big)^{n+1}\times$$
$$\times|z_{13}z_{23}|^2\prod_{i<j}^{3} z_{ij}^{k_i\cdot k_j-\frac{2}{k}(m_im_j-j_ij_j)}\bar{z}_{ij}^{k_i\cdot k_j-\frac{2}{k}(\bar{m}_i\bar{m}_j-j_ij_j)}\times$$
$$\times\int d^{2n+2}y_i d^{2n}y_{\hat{i}}\prod_{i=1}^{n+1}\Big[(y_i-z_1)^{j_1-m_1}(y_i-z_2)^{j_2-m_2}(y_i-z_3)^{j_3-m_3-1}\times$$
$$\times(\bar{y}_i-\bar{z}_1)^{j_1+\bar{m}_1}(\bar{y}_i-\bar{z}_2)^{j_2+\bar{m}_2}(\bar{y}_i-\bar{z}_3)^{j_3+\bar{m}_3-1}\Big]\times$$
$$\times\prod_{\hat{i}=1}^{n}\Big[(y_{\hat{i}}-z_1)^{j_1+m_1}(y_{\hat{i}}-z_2)^{j_2+m_2}(y_{\hat{i}}-z_3)^{j_3+m_3+1}\times$$
$$\times(\bar{y}_{\hat{i}}-\bar{z}_1)^{j_1-\bar{m}_1}(\bar{y}_{\hat{i}}-\bar{z}_2)^{j_2-\bar{m}_2}(\bar{y}_{\hat{i}}-\bar{z}_3)^{j_3-\bar{m}_3+1}\Big]\times$$
$$\times\prod_{i<j}|y_{ij}|^2\prod_{\hat{i}<\hat{j}}|y_{\hat{i}\hat{j}}|^2\prod_{i,\hat{j}}|y_{i\hat{j}}|^{-2-2k}. \tag{5.15}$$

Using the relation between quantum numbers

$$\sum_a m_a = \frac{k}{2},\qquad \sum_a \bar{m}_a = -\frac{k}{2},\qquad \sum_a j_a + 1 = \frac{k}{2}(2n+1), \tag{5.16}$$

we can indeed find that the above expression is independent of $z_a$ as expected from conformal invariance. If we put $z_1 = 0$, $z_2 = 1$ and $z_3 = \infty$, it reduces to the formula (3.27) in the previous section. Ignoring the phase, we finally obtain the following expression:

$$A_+ = \delta^n\Big(\sum_i k_i\Big)\delta\Big(\sum p_a\Big)\delta\Big(\sum w_a - 1\Big)\times$$
$$\times\frac{\pi\mu_+^n\mu_-^{n+1}}{k\sin[(2n+1)\pi]}\frac{\Gamma(1+j_1-m_1)\Gamma(1+j_2-m_2)\Gamma(1+j_3-m_3)}{\Gamma(-j_1-\bar{m}_1)\Gamma(-j_2-\bar{m}_2)\Gamma(-j_3-\bar{m}_3)}\times$$
$$\times\prod_{s=1}^{n}\frac{\Gamma(-sk)\Gamma(1+2j_1-sk)\Gamma(1+2j_2-sk)\Gamma(1+2j_3-sk)}{\Gamma(1+sk)\Gamma(sk-2j_1)\Gamma(sk-2j_2)\Gamma(sk-2j_3)}. \tag{5.17}$$

We see that the three-point function is manifestly symmetric in three vertices. Note that the string coupling was absorbed in the coupling $\mu_\pm$ by an appropriate shift of $\rho$. It can be restored using the standard scaling arguments of [29].

## 6. Conclusion

In this paper we analyzed the three-point functions of tachyons in sine-Liouville theory which is believed to be dual to the coset model $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$. We checked that the sine-Liouville theory correctly reproduces the three-point functions in the coset model when the winding number is conserved. We also derived that the violation of total winding number is up to $\pm 1$ in three-point functions. All these are in agreement with the work of FZZ. We also obtained the expression for generic three-point functions when the winding number is non-conserving.

In the case where the total winding number is $\pm 1$, it seems that the correlators in sine-Liouville theory do not fully reflect the $\mathrm{SL}(2,\mathbb{R})$ group structure, since in general the Clebsch-Gordan coefficients for $\mathrm{SL}(2,\mathbb{R})$ are expressed in terms of the generalized hypergeometric functions ${}_3F_2$. It would be interesting to study what the implication is.

Obviously, the theories studied in this paper is closely related to the $\mathrm{SL}(2,\mathbb{R})$ WZW model which describes a string on $AdS_3$. Our new result will give a new insight into the analysis of the processes involving a "long string" which has a macroscopic configuration of worldsheet winding around the boundary of $AdS_3$. This will be our future problem.

**Note added.** After the first version of this paper was submitted, the authors received some comments from V. Fateev that the same result was obtained in the unpublished work using essentially the same techniques. Their analysis used some integral identities the spinless case of which can be found in [30]. Some comments about this duality can also be found in [31].

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## A. New integral formulae

Let us summarize here our new formulae obtained in this paper.

We calculated the integral of the form

$$I = \frac{1}{m!n!}\int \prod_{i=1}^{m}\left(d^2 z_i z_i^{\alpha}\bar{z}_i^{\bar{\alpha}}(1-z_i)^{\beta}(1-\bar{z}_i)^{\bar{\beta}}\right)\prod_{i<j}^{m}|z_{ij}|^2 \times$$
$$\times \prod_{\hat{i}=\hat{1}}^{n}\left(d^2 z_{\hat{i}} z_{\hat{i}}^{\alpha'}\bar{z}_{\hat{i}}^{\bar{\alpha}'}(1-z_{\hat{i}})^{\beta'}(1-\bar{z}_{\hat{i}})^{\bar{\beta}'}\right)\prod_{\hat{i}<\hat{j}}^{n}|z_{\hat{i}\hat{j}}|^2\prod_{i,\hat{j}}^{m,n}|z_{i\hat{j}}|^{2-2k}\,. \qquad \text{(A.1)}$$

This was found to be nonzero only when $m = n \pm 1$ or $m = n$. Introducing $k' = k - 2$ and the parameters $\gamma$, $\gamma'$, $\bar{\gamma}$ and $\bar{\gamma}'$ satisfying

$$\alpha + \beta + \gamma = \bar{\alpha} + \bar{\beta} + \bar{\gamma} = kn - m - n - 1\,,$$
$$\alpha' + \beta' + \gamma' = \bar{\alpha}' + \bar{\beta}' + \bar{\gamma}' = km - m - n - 1\,,$$

the integral becomes

$$\begin{aligned}
&(m = n+1)\\
&I = -\pi^{m+n}(-)^{\gamma-\bar{\gamma}} \frac{\Gamma(1+\alpha)\Gamma(1+\beta)\Gamma(1+\gamma)}{\Gamma(-\bar{\alpha})\Gamma(-\bar{\beta})\Gamma(-\bar{\gamma})} \times \\
&\quad \times \prod_{j=1}^{n} \frac{\Gamma(-jk')\Gamma(1+\alpha+\alpha'-jk')\Gamma(1+\beta+\beta'-jk')\Gamma(1+\gamma+\gamma'-jk')}{\Gamma(1+jk')\Gamma(jk'-\alpha-\alpha')\Gamma(jk'-\beta-\beta')\Gamma(jk'-\gamma-\gamma')}\,,\\
&(m = n)\\
&I = \pi^{2n-2} I_1 \prod_{j=1}^{n-1} \frac{\Gamma(-jk')\Gamma(1+\alpha+\alpha'-jk')\Gamma(1+\beta+\beta'-jk')\Gamma(1+\gamma+\gamma'-jk')}{\Gamma(1+jk')\Gamma(jk'-\alpha-\alpha')\Gamma(jk'-\beta-\beta')\Gamma(jk'-\gamma-\gamma')}\,,\\
&I_1 = \int d^2z d^2w z^{\alpha}(1-z)^{\beta}\bar{z}^{\bar{\alpha}}(1-\bar{z})^{\bar{\beta}} w^{\alpha'}(1-w)^{\beta'}\bar{w}^{\bar{\alpha}'}(1-\bar{w})^{\bar{\beta}'}|z-w|^{-2-2nk'}\,.
\end{aligned} \tag{A.2}$$

We can rewrite $I_1$ using certain double-contour integrals $c^{ab}$ and $p^{ab}$ as

$$I_1 = c^{12}[\alpha,\beta,\gamma]p^{12}[\bar{\alpha},\bar{\beta},\bar{\gamma}] + c^{21}[\alpha,\beta,\gamma]p^{21}[\bar{\alpha},\bar{\beta},\bar{\gamma}]\,. \tag{A.3}$$

Using the function $G[{}^{a,b,c}_{e,f}] \equiv \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(e)\Gamma(f)}\,{}_3F_2(a,b,c;e,f;1)$, the integral $c^{12}$ is expressed as follows:

$$c^{12} = \frac{\Gamma(1+\alpha+\alpha'-nk')\Gamma(1+\beta+\beta'-nk')}{\Gamma(nk'-\gamma-\gamma')} G\begin{bmatrix} \alpha'+1, \beta+1, nk'-\gamma-\gamma' \\ \alpha'-\gamma+1, \beta-\gamma'+1 \end{bmatrix}, \tag{A.4}$$

and $c^{ab}$ are defined by the permutation of parameters. $p^{ab}$ are related to $c^{ab}$ via

$$\begin{bmatrix} p^{12} \\ p^{21} \end{bmatrix} = A_\beta \begin{bmatrix} c^{23} \\ c^{32} \end{bmatrix} = A_\alpha^T \begin{bmatrix} c^{31} \\ c^{13} \end{bmatrix}, \qquad A_\alpha = \begin{bmatrix} \mathbf{s}(\alpha)\mathbf{s}(\alpha') & -\mathbf{s}(\alpha)\mathbf{s}(\alpha'-nk') \\ -\mathbf{s}(\alpha')\mathbf{s}(\alpha-nk') & \mathbf{s}(\alpha)\mathbf{s}(\alpha') \end{bmatrix}. \tag{A.5}$$

where $\mathbf{s}(x) = \sin(\pi x)$.

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